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SPECTRAL RESPONSE OF A BILINEAR OSCILLATOR

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An approximate analytical procedure is presented to estimate the response power spectral density of a randomly excited spring/mass/damper system having a bilinear spring. The approximate expression for the response spectrum is developed by representing the non-linear oscillator as a linear system having a natural frequency that depends on the envelope of the random response. This approximate representation of the system leads to estimates of the response spectrum that agree extremely well with those obtained by direct numerical simulation of the governing equation.

1. INTRODUCTION

In experimental studies of the response of structures to random excitations it is very common to characterize the response by measuring the power spectral density. Vibration engineers are usually very familiar with this representation of the structural behavior and, at a glance, can gain considerable insight into the system being studied. One can easily see how many resonant modes contribute to the response by looking at the resonant peaks in the spectrum and one can also determine whether the system is heavily or lightly damped. There are situations, however, where non-linear effects in the structure influence the measured power spectrum in a manner that precludes straightforward interpretation. In cases where the response has sufficient amplitude to elicit non-linear behavior, the resonant response peaks in the power spectrum can take a dramatically different form than those observed in linear systems. The main goal of the present investigation is to propose a new approximate scheme for describing the influence of structural non-linearities on the response power spectrum. It is hoped that the present study will provide a better understanding of the random response of non-linear systems.

The present approximate representation of the spectrum is applied to a bilinear oscillator in which the non-linearity has a very pronounced influence on the response spectrum. Comparisons are presented of the estimated power spectral density obtained using the present approximate scheme, and by direct numerical simulation of the governing equation. Excellent agreement is observed between the two methods.

The effect of non-linearities on the response power spectral density has been studied by a number of investigators. Early approximate analytical methods have been presented that are based on either a perturbation approach [1] or equivalent linearization [2, 3]. These methods are found to give reasonable results only for very small non-linearities. Other studies based on numerical simulations have shown that in the case of a Duffing oscillator, where the stiffness contains a cubic non-linearity and the damping is assumed to be linear, the resonant response peak in the power spectral density tends to broaden and increase in frequency as the level of the random excitation is increased [4–6]. This behavior has been observed experimentally in studies of the high level random response of beams and plates [7, 8].

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An approximate analytical procedure was proposed by the author to estimate the response power spectrum of a Duffing oscillator with linear viscous damping [9]. The method is based on an adaptation of the method of equivalent linearization, where the resonant frequency of the equivalent linear system is assumed to be random. This provided very accurate estimates of the spectrum when compared to results obtained using numerical simulations. It has not been possible, however, to extend the method to more general non-linear random systems. The main purpose of the present study is to propose an approximate scheme that may be applicable to a wider class of non-linear systems than studied previously.

The basic assumption of the method proposed in reference [9] is that the response spectrum is strongly influenced by non-linearities in the system, because in a non-linear system the stiffness, and hence the natural frequency, depend on the response amplitude. In a system with sufficiently light damping, the random response will behave as a narrow-band process with an oscillation period that will vary randomly. The random variation of the oscillation period is a direct result of the fact that the response amplitude varies randomly. The random fluctuation of the dominant oscillation frequency will lead to a broadened resonant response peak in the power spectrum. It is reasonable to assume, therefore, that if certain statistics are known concerning the random amplitude fluctuations, and if the relationship between the system resonant frequency and amplitude is known, then one should be able to approximate the response spectrum.

The procedure presented in the following is based on assuming that the non-linear system may be approximated as a linear system having a natural frequency that varies with the response amplitude in the same manner as that predicted when the non-conservative forces are not present. It is assumed that in the damped system with random excitation, the natural frequency depends on the envelope of the response of the original non-linear system. The present method, thus, depends on knowledge of the statistics of the response envelope of the original non-linear system being studied. The probability density of the response envelope is known for several classes of non-linear systems [10]. The random fluctuations in the natural frequency are then assumed to occur much more slowly than the fluctuations in the response. This leads to a simple formula for the response spectrum in the form of an integration over the probability density of the envelope of the response of the original non-linear oscillator.

The present approach has been applied to a bilinear oscillator where the restoring force is assumed to be equal to the deflection (stiffness = 1) for small deflections, and is equal to twice the deflection (stiffness = 2) when the amplitude of the deflection is greater than unity. The damping is assumed to be linear viscous damping. This system is highly non-linear when the random response spends a significant amount of time crossing over between the two stiffness regimes. Numerical simulations of the random response have shown that this bilinear stiffness characteristic has a pronounced effect on the predicted power spectrum. It is hoped that if accurate spectral estimates can be obtained for this rather difficult system, then the approximate method may be applicable to a fairly broad class of oscillators. Comparisons of spectra obtained using the present approximate scheme with those obtained using numerical simulations are presented and show excellent agreement.

As discussed above, the main purpose of the present study is to propose a description of the influence of non-linearities on the power spectra of random structures with random excitation. Although the basic assumptions of the approach are plausible, a detailed error analysis has yet to be conducted. The validity of the approach is investigated here through a single example, the bilinear oscillator. A more detailed investigation of the influence of a number of approximations in the procedure will be performed in a future study.

2. APPROXIMATE REPRESENTATION OF THE NON-LINEAR SYSTEM

Consider a non-linear oscillator governed by

$$\ddot{x}+\omega_0^2(x+\varepsilon g(x))+c\dot{x}=f(t), \tag{1}$$

where $f(t)$ is Gaussian white noise, $g(x)$ is a non-linear restoring force, ω_0 is the natural frequency in the absence of non-linear effects, ε is a constant and c is a viscous damping coefficient. For the present investigation, we consider only systems having conservative non-linearities. The function $g(x)$ does not depend on velocity $\dot{x}$. Non-linear damping effects will be left for future studies. It is also assumed that $g(x)$ is an odd function of x, i.e., $g(x)=-g(-x)$.

To develop a representation of the non-linear random response, consider the behavior of a conservative non-linear system. It is well known that in the case of conservative non-linear systems, where c and $f(t)$ are zero, the solution of equation (1), $x(t)$, will consist of an oscillation having a period that depends on the amplitude of the motion. The exact solution for the period, T, corresponding to an oscillation amplitude, a, is given by

$$T(a)=4\int_0^a\left[2\int_x^a \omega_0^2(u+\varepsilon g(u))\,\mathrm{d}u\right]^{-1/2}\mathrm{d}x. \tag{2}$$

In the case of the damped system with random excitation, described by equation (1), if the damping coefficient c is sufficiently small the response may be considered to be a narrow-band random process. The period of the oscillation cycles can be taken to be the time duration between occurrences of zero crossings with positive slope. The time required for one such cycle will depend on some measure of the oscillation amplitude. This follows from the fact that the stiffness of the non-linear system, and hence the effective natural frequency, depend on the response amplitude.

One can also view the response in the phase plane. In the case of the conservative system where c and $f(t)$ are zero, the amplitude of a given orbit in the phase plane determines the oscillation frequency. If the damping and excitation are sufficiently small, during one cycle the orbit of the non-conservative system will consist of a small fluctuation about that of the conservative system. If the amplitudes of the conservative and non-conservative orbits are nearly the same, the time duration of the cycles will also be similar.

During the forced, damped response of the system described by equation (1), the amplitude of the motion will vary randomly and, based on the above discussion, we can also expect the oscillation period to vary accordingly. An approximate representation of the response of the system described by equation (1) may then be obtained by idealizing it as a linear system having a natural frequency that depends on the response amplitude. This amplitude dependent natural frequency is taken to be identical to that obtained for the conservative system,

$$\omega(a)=2\pi/T(a), \tag{3}$$

where $T(a)$ is given in equation (2). The oscillation frequency is thus regarded as an intrinsic property of the system; it does not depend explicitly on the excitation or the damping as long as the damping is sufficiently light. During the forced, damped response, the oscillation frequency depends only on the amplitude of the motion, as in the case of the conservative system. The amplitude a is a random quantity which, for the present study, is taken to be equal to the envelope of the random response of the original system in equation (1). If $f(t)$ is Gaussian white noise, then the probability density of the envelope and $\mathscr{V}(a)$ is the potential energy of the system corresponding to the response amplitude a,

$$\mathscr{V}(a) = \omega_0^2\left(a^2/2 + \varepsilon \int_0^a g(x)\, \mathrm{d}x\right). \tag{20}$$

σ_0^2 is the mean square response of equation (1) when $\varepsilon = 0$,

$$\sigma_0^2 = \Phi_f \pi/(c\omega_0^2). \tag{21}$$

The approximate response spectrum is then given by

$$\Phi_x(\omega) = \int_0^\infty \frac{\Phi_f}{(\omega^2(a) - \omega^2)^2 + (c\omega)^2} P(a)\, \mathrm{d}a. \tag{22}$$

4. NUMERICAL RESULTS FOR A BILINEAR OSCILLATOR

Equation (22) has been applied to estimate the response spectrum of a bilinear system having a non-linear restoring force as shown in Figure 1. The restoring force shown in Figure 1 is equal to $\omega_0^2(x + \varepsilon g(x))$ in equation (1). In the present calculations, the system is assumed to have a linear force/deflection characteristic with unit slope when the amplitude of the response is less than unity. For greater response amplitudes, the stiffness of the system is assumed to be doubled as shown in the figure. The viscous damping coefficient c in equation (1) is taken to be 0·005 and ω_0 is unity.

The results of applying equation (22) for a range of input force spectrum levels Φ_f are shown in Figures 2(a)–(d). The figures also show the estimated spectra obtained by direct numerical simulation of the response of equation (1). This was computed by calculating the time domain response and estimating the spectrum by a Fast Fourier Transform. The

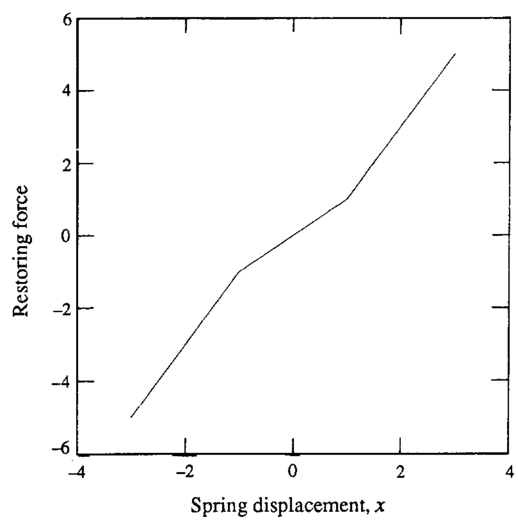

Figure 1. Force-deflection curve of the bilinear spring. The stiffness is assumed to be unity when the deflection amplitude is less than unity. For deflections having magnitude greater than unity the stiffness is assumed to equal two. The restoring force shown is equal to $\omega_0^2(x + \varepsilon g(x))$ in equation (1).

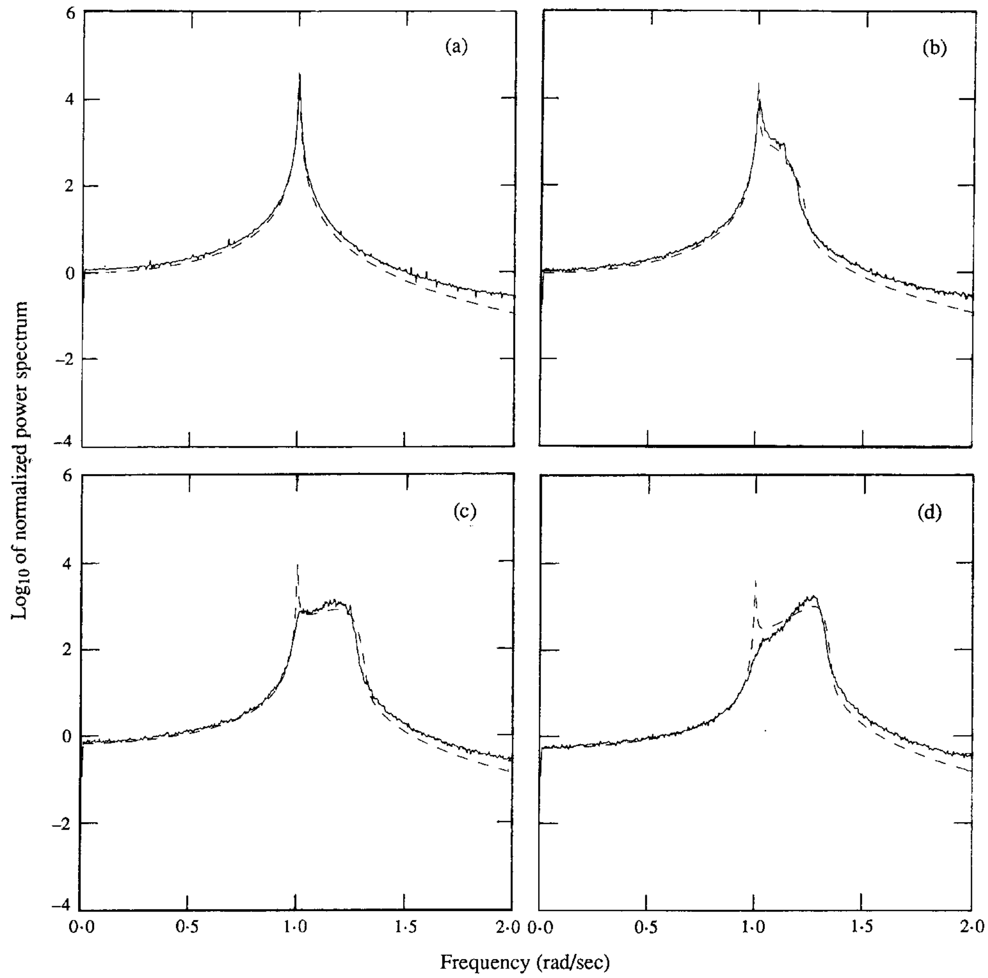

Figure 2. Predicted power spectra for the bilinear system. The results shown are equal to $\log_{10}(\Phi_x(\omega)/\Phi_f)$. The spectra are thus normalized relative to the spectrum of the input signal. Solid curves (——) are the results of time domain simulations, and dashed curves (– – –) are the results of equation (22). The power spectra of the input signal are: (a) $\Phi_f = 0{\cdot}0001$; (b) $\Phi_f = 0{\cdot}001$; (c) $\Phi_f = 0{\cdot}004$, (d) $\Phi_f = 0{\cdot}01$.

figures show that the power spectrum exhibits a broadened shape when the non-linearity is significant. The effect of the non-linear restoring force on the spectrum shape is depicted extremely well by the approximate method of equation (22).

The primary discrepancy appears in Figures 2(c) and (d), where the approximate expression in equation (22) predicts a significant peak at 1 rad/s. Since neither solution is exact, it is desirable to employ a third solution method to examine this effect.

5. CONCLUSIONS

An approximate method has been proposed for estimating the response power spectral density of a non-linear oscillator. The approach is based on considering the non-linear system to behave as a linear system having a randomly varying natural frequency. The natural frequency of the linear system is assumed to depend on the envelope of the non-linear response for which the probability density can be estimated. The result is a simple expression for the response spectrum in the form of an integration over the envelope probability density. Comparisons of results obtained by the present approximate method with those obtained by numerical simulations show excellent agreement.

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